JOURNAL OF DIFFERENTIAL EQUATIONS 65, 240–249 (1986)

The Hölder Property in Some Degenerate Parabolic Problems*

GASTON E. HERNANDEZ

Facultad de Matemáticas,
Pontificia Universidad Católica de Chile, Casilla 6177, Santiago, Chile

Received September 24, 1984; revised July 31, 1985

In this paper it is proved that a solution of $uu_{xx}+u_x^2+h(x,t,u)u=u_t$ with non-negative initial data $u_0(x)$, is α-Hölder continuous for any $\alpha\in(0,1)$, with constant K independent of the modulus of continuity of h. Here u is the limit of solutions u_ε to the problems $(uu_x)_x+h(x,t,u)(u-\varepsilon)=u_t$, $u(x,0)=u_0(x)+\varepsilon$. We prove that $|u|_\alpha\leqslant K$, independent of ε and the modulus of continuity of h. A similar result holds for the general porous medium equation $u_t=(u^m)_{xx}+h(x,t,u)u$, for any $\alpha\in(0,1/(m-1))$. © 1986 Academic Press, Inc.

INTRODUCTION

To prove existence of solutions of some degenerate parabolic equations like

$$uu_{xx}+u_x^2+h(x,t,u)u=u_t \tag{1}$$

that arise in population dynamics problems (see Refs. [5–7] with $h=\beta(u)\,q(x,t)-\mu(u)$) or gas through porous medium (with $h\equiv 0$), with given initial data $u_0(x)\geqslant 0$, $h(x,t,u)$ continuous and bounded in $\Omega_T=\mathbb{R}\times[0,T]$, we can define approximating solutions u_ε of

$$\begin{aligned} E(u)u_{xx}+u_x^2+h(x,t,u)(u-\varepsilon)&=u_t\\ u(x,0)&=u_0(x)+\varepsilon, \end{aligned} \tag{2}$$

where $E(u)$ is a C^∞ function that equals u for $u\geqslant\varepsilon$, equals $\varepsilon/2$ for $u\leqslant\varepsilon/2$ and it increases from $\varepsilon/2$ to ε in $[\varepsilon/2,\varepsilon]$, and then let ε tend to 0. To get convergence of these solutions we need to have some type of a priori estimates for u_x or some regularity of u. These estimates should be indepen-

* Research supported by project 25/85 of the Dirección de Investigación de la Pontificia Universidad Católica de Chile.

0022-0396/86 $3.00

dent of ε, and in some cases, like in Ref. [7], it is necessary that they do not depend on the modulus of continuity of h.

In this paper we prove that if u is a classical solution of (1) with $u \geqslant \varepsilon$, then for any $\alpha \in (0, 1)$, the α-norm of u

$$|u|_\alpha = \sup_{\Omega_T} \frac{|u(x, t) - u(y, s)|}{|x - y|^\alpha + |t - s|^{\alpha/2}}$$

can be estimated in terms of only T, α, the initial function $u_0(x)$ and M_1, the maximum of h. (Thus, if we let ε tend to 0, we can get a limiting function that will be a (weak) solution of (1).)

The case $h = 0$ corresponds to the porous media equation. There is an extensive literature about it (see, for instance [8]). Aronson in [1] showed that u is Lipshitz continuous in x, in one dimension. Gilding in [4] has shown that u is also Hölder continuous in t with exponent $\frac{1}{2}$. Caffarelli and Friedman [2] proved that u is Hölder continuous in higher dimensions.

Related results are also given by Di Benedetto [3] when $h \neq 0$ and in higher dimensions.

In what follows we assume that $h(x, t, u)$ is a continuous function bounded by M_1, $u_0(x)$ is a Lipchitz continuous non-negative function supported in $(-x_1, x_1)$, with $|u_0(x), u_0'(x)| \leqslant M_0$.

We note that if u is a solution of (2), $u \in C^{2,1}(\Omega_T)$, $u \leqslant M$, then by the maximum principle $u \geqslant \varepsilon$. So u is also a solution of

$$\begin{aligned} uu_{xx} + u_x^2 + h(x, t, u)(u - \varepsilon) &= u_t \\ u(x, 0) &= u_0(x) + \varepsilon \end{aligned} \tag{2'}$$

and, in fact, it is the unique solution in $C^{2,1}(\Omega_T)$ which satisfies $u \geqslant \varepsilon$.

LEMMA 1. *Assume that $u \in C^{2,1}(\Omega_T)$, $u \leqslant M$ and u is a solution of (2'). Then the oscillation of u tends to 0 as $|x|$ tends to ∞.*

Proof. From the maximum principle we get that $u \geqslant \varepsilon$. We will show that u lies below an appropriate barrier function $\varepsilon v(x, t)$, with $v(x, t)$ tending to 1 as $|x|$ goes to ∞. To that effect we let $m \geqslant \max(3, M_1/2M\}$, $t_2 = (4M_1)^{-1}$, $x_2 \geqslant x_1 + m$ and define

$$v(x, t) = \frac{(x - x_2)^2}{4m(2t_2 - t)} + \varepsilon \left[\frac{2t_2}{2t_2 - t}\right]^{1/m}.$$

If $L[z] = E(z)z_{xx} + z_x^2 + h(x, t, u(x, t))(u - \varepsilon) - z_t$ we have that by the choices of m and t_2, $L[v] \leqslant 0 \leqslant L[u]$ in $\Omega = (x_2 - m, x_2 + m) \times (0, t_2)$ and

$u \leqslant v$ on the boundary of Ω. Thus by the maximum principle $u(x, t) \leqslant v(x, t)$ in Ω. In particular

$$u(x_2, t) \leqslant \varepsilon \left[\frac{2t_2}{2t_2 - t}\right]^{1/m} \leqslant 2^{1/m}\varepsilon \qquad \text{for} \quad t \in [0, t_2].$$

Since the only condition for x_2 is to be larger than $x_1 + m$, it follows that $u(x, t) \leqslant 2^{1/m}\varepsilon$ for all $x \geqslant x_1 + m$, $t \in [0, t_2]$. We let $k = 1 + [T/t_2]$, $x_k = x_1 + km$, then in at most k steps we obtain tat $u(x, t) \leqslant 2^{k/m}\varepsilon$ for $x \geqslant x_k$, $t \in [0, T]$. The same argument is valid for $x \leqslant -x_k$, so $\varepsilon \leqslant u(x, t) \leqslant 2^{k/m}\varepsilon$ for $|x| \geqslant x_k$, $t \in [0, T]$. Now, given $\delta > 0$, since k is fixed, we can choose m large, depending on δ, such that $2^{k/m} \leqslant 1 + \delta/\varepsilon$; hence, by the above argument $\varepsilon \leqslant u \leqslant 2^{k/m}\varepsilon \leqslant \varepsilon + \delta$ for $|x| \geqslant x_1 + km$.

The Hölder property

THEOREM 1. *Let $u \in C^{2,1}(\Omega_T)$ be a solution of (2′) with $|h| \leqslant M_1$, $0 < \varepsilon \leqslant u \leqslant M$, $|u_0, u_0'| \leqslant M_0$. Assume that the first derivatives of u are bounded in Ω_T. Then there exists a constant K_1 depending only on M_0, M_1, T and α such that $|u|_\alpha \leqslant K_1$ in Ω_T. In particular K_1 is independent of ε and the modulus of continuity of h.*

Proof. Let $\delta > 0$ be a small number. By Lemma 1, there exists B_δ such that $|u(x, t) - u(y, s)| \leqslant \delta$ for $|x|$, $|y| \geqslant B_\delta$. Let $A_\delta = \max\{1/\delta, B_\delta + 1\}$, $\lambda = 2/\alpha$,

$$W_{1,\delta} = [-A_\delta, A_\delta] \times [-A_\delta, A_\delta] \times [0, T] \times [0, T],$$
$$W_{2,\delta} = \{(x, y, t, s) \in W_{1,\delta}/|x - y| \geqslant \delta \text{ or } |t - s| \geqslant \delta\}$$

and define

$$g(x, y, t, s) = \frac{|u(x, t) - u(y, s)|^\lambda}{(x - y)^2 + K|t - s|}$$

if $x \neq y$ or $s \neq t$ and $g(x, x, t, t) = 0$. K is a constant to be determined later.

The function g thus defined is continuous in $W_{1,\delta}$. To verify this, it is enough to check at the points $x = y$, $t = s$. Since u_x and u_t are supposed to be bounded, the mean value theorem and the inequality $(a + b)^p \geqslant 2^{p-1}(a^p + b^p)$, for $a, b > 0$, $0 < p < 1$, imply that

$$g^{\alpha/2}(x, y, t, s) \leqslant \frac{|u(x, t) - u(y, s)|}{2^{\alpha/2 - 1}(|x - y|^\alpha + K^{2/\alpha}|t - s|^{\alpha/2})},$$

so

$$g^{\alpha/2}(x, y, t, s) \leqslant 2(|u_x|_\infty |x - y|^{1-\alpha} + |u_t|_\infty |t - s|^{1-\alpha/2}). \tag{3}$$

(Assuming $K \geqslant 1$.) This implies that g is continuous at $x = y$, $t = s$.

We also assume that δ is so small that $|u_x|_\infty \leqslant \delta^{\alpha-1}$ and $K^{\alpha/2}|u_t|_\infty \leqslant \delta^{(\alpha-2)/2}$. We will show that $g \leqslant K_2$ in $W_{2,\delta}$, then since the domains $W_{2,\delta}$ converge to the domain $\{(x, y, t, s)/x \neq y \text{ or } t \neq s\}$ as δ goes to 0, and g is continuous in $R^2 \times [0, T]^2$, we conclude that $g \leqslant K_2$ in $R^2 \times [0, T]^2$. This implies the theorem with $K_1 = (K_2 K)^{\alpha/2}$.

Clearly g is continuous in $W_{2,\delta}$. Let us assume that g attains its maximum at a point $Q_1 = (x_1, y_1, t_1, s_1)$. We have the following possibilities:

(a) g is not differentiable at Q_1 (i.e., $t_1 = s_1$ or $g = 0$).

(b) Q_1 is on the lateral boundaries of $W_{2,\delta}$ (i.e. $|x_1| = A_\delta$ or $|y_1| = A_\delta$).

(c) Q_1 is on the "interior" boundaries of $W_{2,\delta}$ ($|x_1 - y_1| = \delta$ and $|t_1 - s_1| \leqslant \delta$ or $|x_1 - y_1| \leqslant \delta$ and $|t_1 - s_1| = \delta$).

(d) Q_1 is on the lower boundaries of $W_{2,\delta}$ (i.e., $t_1 = 0$ or $s_1 = 0$).

(e) Q_1 is an interior point of $W_{2,\delta}$ at which g is differentiable.

Now we prove two lemmas that treat these cases separately and then we go back to the main proof.

LEMMA 2. *Let*

$$\Omega_{3,\delta} = \{(x, y, t)/|x|, |y| \leqslant A_\delta, |x - y| \geqslant \delta\},$$

$$k(x, y, t) = \frac{|u(x, t) - u(y, t)|^\lambda}{(x-y)^2} = \frac{|S|^\lambda}{R^2}.$$

Then k is bounded in $W_{3,\delta}$.

Proof. Suppose that $\max k$ occurs at $Q_0 = (x_0, y_0, t_0)$. We have the following possibilities:

(a) $t_0 = 0$.

(b) $|x_0 - y_0| = \delta$.

(c) $|x_0| = A$ or $|y_0| = A$.

(d) Q_0 is an interior point of $\Omega_{3,\delta}$ at which k is differentiable.

If the first case occurs,

$$k(x_0, y_0, 0) = \frac{|u(x_0, 0) - u(y_0, 0)|^\lambda}{(x_0 - y_0)^2} \leqslant (M_0)^\lambda.$$

If the second case occurs,

$$\begin{aligned} k(x_0, y_0, t_0) &= \delta^{-2}\,|u_1 - u_2|^\lambda \\ &\leqslant \delta^{-2}\,|u_x|^\lambda\,\delta^\lambda \\ &\leqslant \delta^{-2}(\delta^{\alpha-1})^\lambda\,\delta^\lambda = \delta^{\alpha\lambda-2} = 1. \end{aligned}$$

Suppose the third case occurs. Assume $|x_0| = A_\delta$.
If $|y_0| \leqslant A_\delta - 1$ then $|x_0 - y_0| \geqslant 1$, so $k \leqslant (2M)^\lambda$.
If $A_\delta - 1 \leqslant |y_0| \leqslant A_\delta$ then $\mathrm{osc}(u) \leqslant \delta$, so $k \leqslant \delta^{\lambda-2} \leqslant 1$.
If the last case occurs, since k is differentiable at Q_0, we have $k_x = k_y = 0$ and k_{xx}, k_{yy}, $-k_t \leqslant 0$ there. We write u_1 for $u(x, t)$ and u_2 for $u(y, t)$. Then

$$u_1 k_{xx} + u_2 k_{yy} - k_t \leqslant 0, \qquad \text{at } Q_0.$$

The derivatives are:

$$\begin{aligned} k_x &= \lambda\,|S|^{\lambda-1} u_{1x} R^{-2}\sigma - 2\,|S|^\lambda\,R^{-3} \qquad (\sigma = \mathrm{sgn}(S)) \\ k_y &= -\lambda\,|S|^{\lambda-1} u_{2y} R^{-2}\sigma + 2\,|S|^\lambda\,R^{-3} \\ k_{xx} &= \lambda\,|S|^{\lambda-1} u_{1xx} R^{-2}\sigma + \lambda(\lambda-1)\,|S|^{\lambda-2} u_{1x}^2 R^{-2} \\ &\quad - 4\lambda\,|S|^{\lambda-1} u_{1x} R^{-3}\sigma + 6\,|S|^\lambda R^{-4} \\ k_{yy} &= -\lambda\,|S|^{\lambda-1} u_{2yy} R^{-2}\sigma + \lambda(\lambda-1\,|S|^{\lambda-2} u_{2y}^2 R^{-2} \\ &\quad - 4\lambda\,|S|^{\lambda-1} u_{2y} R^{-3}\sigma + 6\,|S|^\lambda R^{-4} \\ k_t &= \lambda\,|S|^{\lambda-1} R^{-2}\sigma(u_{1t} - u_{2t}). \end{aligned}$$

Replacing these values in (4) we obtain

$$\begin{aligned} &\lambda\,|S|^{\lambda-1} R^{-2}\sigma[u_1 u_{1xx} - u_2 u_{2yy} - u_{1t} + u_{2t}] \\ &\quad + \lambda(\lambda-1)\,|S|^{\lambda-2} R^{-2}[u_1 u_{1x}^2 + u_2 u_{2y}^2] \\ &\quad - 4\lambda\,|S|^{\lambda-1} R^{-3}\sigma[u_1 u_{1x} + u_2 u_{2y}] \\ &\quad + 6\,|S|^\lambda R^{-4}(u_1 + u_2) \leqslant 0. \end{aligned} \tag{5}$$

In the first term we use the differential equation (2′) in (x, t) and in (y, t), obtaining

$$\lambda\,|S|^{\lambda-1} R^{-2}\sigma(-u_{1x}^2 - h_1(u_1 - \varepsilon) + u_{2y}^2 + h_2(u_2 - \varepsilon)),$$

where $h_i = h(x, y, u_i)$, $i = 1, 2$.

Next, $k_x = k_y = 0$ implies

$$u_{1x} = \alpha\, |S|\, R^{-1}\sigma = u_{2y} \tag{6}$$

so the previous expression becomes $\lambda\, |S|^{\lambda-1}R^{-2}\sigma[h_2(u_2-\varepsilon)-h_1(u_1-\varepsilon)]$.

Using also equality (6) in the second and third terms of (5) we finally obtain:

$$\lambda |S|^{\lambda-1}R^{-2}\sigma[h_2(u_2-\varepsilon)-h_1(u_1-\varepsilon)] + |S|^{\lambda}R^{-4}(u_1+u_2)(2\alpha(\lambda-1)-2)\leqslant 0.$$

Therefore

$$|S|^{\lambda}\, R^{-2} \leqslant \frac{\lambda\, |S|^{\lambda-1}}{2(1-\alpha)}\, \frac{|h_2(u_2-\varepsilon)-h_1(u_1-\varepsilon)|}{u_1+u_2}. \tag{7}$$

Since $|h_i| \leqslant M_1$, one has $|h_i(u_i-\varepsilon)| \leqslant M_1 u_i$, $i=1, 2$. Thus

$$\frac{|h_2(u_2-\varepsilon)-h_1(u_1-\varepsilon)|}{u_1+u_2} \leqslant M_1 \frac{u_1+u_2}{u_1+u_2} = M_1.$$

Since S is bounded by $2M$, we conclude that

$$k(Q_0) = \frac{|S|^{\lambda}}{R^2} \leqslant \frac{\lambda\, |S|^{\lambda-1}M_1}{2(1-\alpha)} \leqslant \frac{(2M)^{\lambda-1}M_1}{\alpha(1-\alpha)},$$

LEMMA 3. *Let*

$$\Omega_{4,\delta} = \{(x, y, t)/|x|, |y| \leqslant A_{\delta}, |x-y| \geqslant \delta \text{ or } t \geqslant \delta\}$$

$$f(x, y, t) = \frac{|u(x, t)-u(y, 0)|^{\lambda}}{(x-y)^2+Kt} = \frac{|S|^{\lambda}}{R}.$$

Then f is bounded in $\Omega_{4,\delta}$.

Proof. The proof is very similar to that of Lemma 2. If $f(x_2, y_2, t_2)$ is maximum, the cases

(a) $t_2 = 0$,

(b) $|x_2 - y_2| = \delta$ and $t_2 \leqslant \delta$,

(c) $|x_2 - y_2| \leqslant \delta$ and $t_2 = \delta$, and

(d) $|x_2| = A$ or $|y_2| = A$

are treated as before, giving that $f(Q_2)$ is bounded by $(M_0)^{\lambda}$, 2^{λ}, 2^{λ} and $1+(2M)^{\lambda}$, respectively.

If (x_2, y_2, t_2) is an interior point, we have that $u_1 f_{xx} + u_2 f_{yy} - f_t \leqslant 0$ and $f_x = f_y = 0$, at $Q_2 = (x_2, y_2, y_2)$.

After replacing the derivatives and dropping the positive terms that contain u_{1x}^2 and u_{2x}^2 we have

$$\lambda\,|S|^{\lambda-1}R^{-1}\sigma[u_1 u_{1xx} - u_2 u_{2yy} - u_{2t}] + 2\,|S|^{\lambda}R^{-2}[K/2 - (u_1+u_2)] \leqslant 0.$$

Using the differential equation in $u(x, t)$ only

$$2\,|S|^{\lambda}R^{-2}[K/2-(u_1+u_2)] \leqslant \lambda\,|S|^{\lambda-1}R^{-1}\sigma[u_{1x}^2 + h_1(u_1-\varepsilon) + u_2 u_{2yy}].$$

This time, $f_x = f_y = 0$ implies $u_{1x} = \alpha(x-y)\,|S|\,R^{-1} = u_{2y}$, so

$$\begin{aligned} &2\,|S|^{\lambda}R^{-2}[K/2-(u_1+u_2)] \\ &\quad\leqslant \lambda\,|S|^{\lambda-1}R^{-1}\,|\alpha^2(x-y)^2\,|S|^2R^{-2} + h_1(u_1-\varepsilon) + u_2u_{2yy}|. \end{aligned}$$

Using that $(x-y)^2R^{-1} \leqslant 1$ and $|S|^{\lambda+1} = |S|^{\lambda}\,|u_2-u_1| \leqslant M\,|S|^{\lambda}$, we find

$$\begin{aligned} 2\,|S|^{\lambda}R^{-1}[K/2-(u_1+u_2+M\alpha)] &\leqslant \lambda\,|S|^{\lambda-1}\,|h_1(u_1-\varepsilon)+u_2u_{2yy}| \\ &\leqslant \lambda M^{\lambda-1}M(M_1+M_0). \end{aligned} \tag{8}$$

Thus, for $K = 8M$ we obtain

$$\frac{|S|^{\lambda}}{R} \leqslant \frac{M^{\lambda-1}}{\alpha}(M_1+M_0).$$

Thus

$$f(x, y, t) \leqslant \max\left\{(M_0)^{\lambda}, 2^{\lambda}, 1+(2M)^{\lambda}, \frac{(2M)^{\lambda-1}}{\alpha}(M_1+M_0)\right\}.$$

After these lemmas we go back to the main proof. If $g(Q_1)$ is maximum, either

$$\text{(a)}\quad t_1 = s_1 \qquad \text{or} \qquad \text{(b)}\quad t_1 \neq s_1.$$

If $t_1 = s_1$, then

$$g(Q_1) = k(x_1, y_1, t_1) \leqslant \frac{(2M)^{-1}M_1}{\alpha(1-\alpha)}.$$

Thus, we may assume that $t_1 \neq s_1$. We have two possibilities:

$$\begin{aligned} &\text{(c)}\quad (|x_1-y_1| = \delta \quad \text{and} \quad |t_1-s_1| \leqslant \delta) \quad \text{or} \\ &\qquad\;\; (|x_1-y| \leqslant \delta \quad \text{and} \quad |t_1-s_1| = \delta) \end{aligned}$$

and

$$\text{(d)} \quad (|x_1 - y_1| > \delta \qquad \text{or} \qquad |(t_1 - s_1| > \delta).$$

If case (c) occurs, by inequality (3) and the assumptions on δ,

$$g^{\alpha/2}(Q_1) \leqslant |u_x|_\infty \, \delta^{1-\alpha} + K^{\alpha/2} \, |u_t|_\infty \, \delta^{1-\alpha/2} \leqslant 1 + 1 = 2.$$

Thus

$$g(Q_1) \leqslant 2^{2/\alpha}.$$

Let us suppose now that $|x_1 - y_1| > \delta$ or $|t_1 - s_1| > \delta$. Here we have three possibilities:

(e) $|x_1| = A$ or $|y_1| = A$,

(f) $t_1 = 0$ or $s_1 = 0$, and

(g) $t_1 > 0$, $s_1 > 0$, $|x_1| < A$ and $|y_1| < A$.

We treat case (e) exactly as in Lemma 2, obtaining $g(Q_1) \leqslant 1 + (2M)^\lambda$. If case (f) occurs, then $g(Q_1) = f(x_1, y_1, t_1) \leqslant ((2M)^{\lambda-1}/\alpha)(M_1 + M_0)$. Finally, if case (g) occurs we have that $g_x = g_y = 0$ and $g_{xx}, g_{yy}, -g_t, -g_s \leqslant 0$ at Q_1. If $s_1 < t_1$ we take the combination $2u_1 \, g_{xx} + u_2 g_{yy} - (2g_t + g_s) \leqslant 0$, introduce in it the differential equation (2′), and choosing $k = 12M$ we obtain

$$\frac{|S|^\lambda}{R} \leqslant \frac{2(2M)^{\lambda-1} M_1}{\alpha}.$$

If $s_1 > t_1$ the same result follows by taking

$$u_1 \, g_{xx} + 2u_2 \, g_{yy} - (g_t + 2g_s) \leqslant 0.$$

Finally, a weak solution u of (1) with $u(x, 0) = u_0(x)$ can be obtained as the limit of a subsequence $\{u_{\varepsilon_K}\}$ of solutions of (2′), that convergences uniformly on compact sets to u, so $|u|_\alpha \leqslant K$.

The previous method can be applied to a wide class of equations where the maximum principle holds and where the oscillation of u tends to 0 as $|x| \to \infty$, or when the domain is finite and the Hölder quotient can be controlled a priori on the boundaries of Ω_T. In particular, the more general porous medium equation

$$\begin{aligned} u_t &= (u^m)_{xx} + h(x, t, u)u, \qquad m \geqslant 2, \\ u(x, 0) &= u_0(x) \geqslant 0 \end{aligned} \tag{9}$$

can be treated with a similar argument.

The change of dependent variable $v = u^{m-1}$ leads to

$$v_t = mvv_{xx} + \frac{m}{m-1} v_x^2 + h(x, t, v^{1/(m-1)}) v^{1/(m-1)},$$
$$v(x, 0) = u_0^{1/(m-1)}(x). \tag{10}$$

The corresponding ε-approximations with $\eta = 1/(m-1)$ are

$$v_t = mvv_{xx} + \frac{m}{m-1} v_x^2 + h(x, t, v^\eta)(v^\eta - \varepsilon^\eta),$$
$$v(x, 0) = u_0^\eta(x) + \varepsilon. \tag{11}$$

In this case we have the following theorem:

THEOREM 2. *Let $v \in C^2(\Omega_T)$ be a solution of* (11). *Let $|h| \leqslant M_1$, $0 < \varepsilon \leqslant v \leqslant M$, $|v_0, v_0'| \leqslant M_0$. Assume that the first derivatives of v are bounded in Ω_T and that* $\operatorname{osc}(v) \to 0$ *as $|x| \to \infty$. Then there exists K_2 constant such that $|v|_\alpha \leqslant K_2$ for any $\alpha \in (0, 1)$. K_2 depends only on M_0, M_1, T and α.*

Proof. The proof follows the same lines of the previous theorem. The only differences are:

At the interior point where k is differentiable, we take $mv_1 k_{xx} + mv_2 k_{yy} - k_t \leqslant 0$.

Then instead of (7) we obtain

$$|S|^\lambda R^{-2} \leqslant \frac{\lambda\, |S|^{\lambda-1}\, |h_2(v_2^\eta - \varepsilon^\eta) - h_1(v_1^\eta - \varepsilon^\eta)|}{2(1-\alpha)\, m(v_1 + v_2)}. \tag{12}$$

Since $\lambda > 2$ we can write

$$|S|^\lambda R^{-2} \leqslant \frac{\lambda\, |S|^{\lambda-2}}{2m(1-k)} \cdot 2M_1 M^{1/(m-1)} \cdot \frac{|v_1 - v_2|}{v_1 + v_2}, \tag{13}$$

which is bounded by

$$\frac{\lambda M^{\lambda-2}}{m(1-\alpha)} M_1 M^{1/(m-1)}.$$

Similarly for the function f_1, instead of (8) we obtain

$$2\, |S|^\lambda R^{-1}[K/2 - (m(v_1 + v_2) + M_\alpha)] \leqslant \lambda M^{\lambda-1} M(M_1 + M_0) \tag{14}$$

and this time we choose $K = 4M(m+1)$.

A similar result holds for the function g.

Now, for the function u we have:

$$\frac{|u(x,t)-u(y,s)|}{|x-y|^{\alpha\eta}+|t-s|^{\alpha\eta/2}} \leqslant \frac{|v^{\eta}(x,t)-v^{\eta}(y,s)|}{|v(x,t)-v(y,s)|^{\eta}} \cdot \frac{|v(x,t)-v(y,s)|^{\eta}}{|x-y|^{\alpha\eta}+|t-s|^{\alpha\eta/2}}$$

$$\leqslant 1 \cdot \left[\frac{|v(x,t)-v(y,s)|}{|x-y|^{\alpha}+|t-s|^{\alpha/2}}\right]^{\eta} \leqslant K_2^{\eta}.$$

That is to say, u is ν-Hölder continuous, for any $\nu \in (0, 1/(m-1))$.

References

1. D. G. Aronson, Regularity properties of flows through porous media, *SIAM J. Appl. Math.* **17** (1969), 461–467.
2. L. A. Caffarelli and A. Friedman, Regularity of the free boundary of a gas flow in an n-dimensional porous medium, *Indiana Univ. Math. J.* **29** (1980), 361–391.
3. E. Di Benedetto, Continuity of weak solutions to a general porous medium equation, *Indiana Univ. Math. J.* **32** (1983), 83–117.
4. B. H. Gilding, Hölder continuity of solutions of parabolic equations, *J. London Math. Soc.* (E) **13** (1976), 103–106.
5. M. E. Gurtin, Some questions and open problems in continuum mechanics and population dynamics, *J. Differential Equations* **48** (1983), 293–312.
6. G. E. Hernandez, Existence of solutions of population dynamics problems with diffusion, thesis. University of Minnesota, 1983.
7. G. E. Hernandez, Existence of solutions in a population dynamics problem, *Quart. Appl. Math.*, in press.
8. L. A. Peletier, The porous media equation, *in* "Applications of Non-linear Analysis in the Physical Sciences" (H. Amann *et al.*, Eds.), pp. 229–241, Pitman, London, 1981.